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Bosonization and Current Algebra of Spinning Strings

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ABSTRACT

We write down a general geometric action principle for spinning strings in d -dimensional Minkowski space, which is formulated without the use of Grassmann coordinates. Instead, it is constructed in terms of the pull-back of a left invariant Maurer-Cartan form on the d -dimensional Poincaré group to the world sheet. The system contains some interesting special cases. Among them are the Nambu string (as well as, null and tachyonic strings) where the spin vanishes, and also the case of a string with a spin current - but no momentum current. We find the general form for the Virasoro generators, and show that they are first class constraints in the Hamiltonian formulation of the theory. The current algebra associated with the momentum and angular momentum densities are shown, in general, to contain rather complicated anomaly terms which obstruct quantization. As expected, the anomalies vanish when one specializes to the case of the Nambu string, and there one simply recovers the algebra associated with the Poincaré loop group. We speculate that there exist other cases where the anomalies vanish, and that these cases give the bosonization of the known pseudoclassical formulations of spinning strings.

1 Introduction

The classical spin of relativistic particles can be described using either classical or pseudoclassical variables.[1] The same result also holds for the classical spin of relativistic strings. Of course, the pseudoclassical description of spinning strings is well known. Descriptions of spinning strings in terms of classical variables were examined recently in ref. [2]. There we looked at the case of $2+1$ space-time only, and the appropriate classical variables took values in the $2+1$ Poincaré group $ISO(2,1)$. The string action was constructed in terms of the pull-back of a left invariant Maurer-Cartan form on $ISO(2,1)$ to the world sheet. Although, it has a particularly elegant form in $2+1$ dimensions due to the existence of a nondegenerate scalar product on the Poincaré algebra $\underline{ISO}(2,1)$ [3], the action can be generalized to the case of an arbitrary number of space-time dimensions, as well as to the case of membranes and p-branes. In this article, we shall examine such generalizations.

As well as generalizing the system of ref. [2] to higher dimensions, the approach described here generalizes that developed by Balachandran, Lizzi and Sparano[4], which in turn gives a unifying description of the Nambu, null and tachyonic strings. We are able to recover the dynamics of ref. [4] when we specialize to the case of spinless strings.¹ Our system also contains the case of pure spin, where there is a nonvanishing spin current, but no momentum current. The existence of different cases is due to the different choices available for the various constants present in the action. These constants are the analogues of the mass and spin for the relativistic particle.

We write down the spinning string action in Sec 2. Our criteria is that it be invariant under Poincaré transformations, as well as, under diffeomorphisms. The result is a straight-

¹Spinning strings were also considered in [4] using a Wess-Zumino term. Here we shall show that there are more possibilities for introducing spin.

forward generalization of the spinning particle action described in [1]. The resulting classical dynamics is obtained in Sec. 3 for four separate cases. These cases include the spinless string of ref. [4], the case of pure spin, as well as the most general spinning string.

As is usual, the Hamiltonian description of the string system contains constraints. We proceed with the constrained Hamiltonian dynamics in Sec. 4. There we write down the general expression for the diffeomorphism generators on a fixed time slice of the string world sheet, and we show that it satisfies the Virasoro algebra. We then compute the current algebra for the momentum and angular momentum densities for the four cases mentioned above. For the case of the spinless string corresponding to ref. [4], we recover the algebra associated with the Poincaré loop group. On the other hand, for the case of pure spin, we get an extension of the Lorentz loop group algebra. The extension consists of complicated anomalous terms which are obstructions to the quantization. Similar results are obtained for the most general spinning string. There we, instead, get an extension of the Poincaré loop group algebra.

If we demand that the above mentioned anomalous terms vanish for quantization, we arrive at a set of equations for the constants defining the system. These equations are quite complicated and we have not yet found their solutions. We speculate that solutions exist and they correspond to the bosonic formulation of known pseudoclassical descriptions of spinning strings. We note that the vanishing of the anomalous terms of the classical Poincaré loop group algebra is a necessary but not sufficient condition for a consistent quantization, as new anomalies are expected to arise at the quantum level. This indeed is known to be the case for spinless strings. We also note that a BRST approach to the quantization of this system does not appear to be straightforward due to the appearance of second class as well as first class constraints in the Hamiltonian formalism.

It is straightforward to generalize our action for spinning strings to higher dimensional spinning objects, like membranes. We show how to do this in Sec. 5.

2 Classical Actions

Before we write down the general expression for the spinning string action, we give a discussion of our mathematical conventions and a brief review of the classical description for the relativistic spinning particle.

2.1 Mathematical Conventions

As stated above, the target space for the spinning string will be taken to be the Poincaré group. We denote an element of the Poincaré group in $d + 1$ space-time dimensions by $g = (\Lambda, x)$, where $\Lambda = \{\Lambda^i{}_j\}$ is a Lorentz matrix and $x = \{x^i\}$ a Lorentz vector, $i, j, \dots = 0, 1, 2, \dots, n$. x will also serve to denote the Minkowski coordinates of the string. We will only consider closed strings so Λ and x are functions on $R^1 \times S^1$, R^1 being associated with the time.

Under the left action of the Poincaré group by $h = (\theta, y)$, g transforms according to the semidirect product rule:

$$(\Lambda, x) \rightarrow (\theta, y) \circ (\Lambda, x) = (\theta\Lambda, \theta x + y) . \quad (1)$$

Similarly, under the right action of the Poincaré group by h , we have that

$$(\Lambda, x) \rightarrow (\Lambda, x) \circ (\theta, y) = (\Lambda\theta, \Lambda y + x) . \quad (2)$$

Let $t_{ij} = -t_{ji}$ and u_i denote a basis for the corresponding Poincaré algebra. For their commutation relations we have

$$\begin{aligned}[t_{ij}, t_{k\ell}] &= \eta_{ik}t_{j\ell} + \eta_{jk}t_{\ell i} + \eta_{i\ell}t_{kj} + \eta_{j\ell}t_{ik}, \\ [t_{ij}, u_k] &= \eta_{ik}u_j - \eta_{jk}u_i, \\ [u_i, u_j] &= 0,\end{aligned}\tag{3}$$

where $[\eta_{ij}] = \text{diag}(-1, 1, 1, \dots, 1)$ is the Minkowski metric.

A left invariant Maurer-Cartan form associated with the Poincaré group can be written as follows:

$$g^{-1}dg = W^{ij}t_{ij} + V^i u_i,\tag{4}$$

where the components W^{ij} and V^i are one forms given by

$$W = \Lambda^{-1}d\Lambda \quad \text{and} \quad V = \Lambda^{-1}dx.\tag{5}$$

It is easy to check that $g^{-1}dg$ is invariant under left transformations (1). Under right transformations (2), the Maurer-Cartan form transforms as follows:

$$g^{-1}dg \rightarrow g'^{-1}dg' = W^{ij} t'_{ij} + V^i u'_i\tag{6}$$

where

$$\begin{aligned}t'_{ij} &= (\theta t \theta^{-1})_{ij} + \frac{1}{2}(\theta u)_i y_j - \frac{1}{2}(\theta u)_j y_i, \\ u'_i &= (\theta u)_i.\end{aligned}\tag{7}$$

The transformation from t_{ij} and u_i to t'_{ij} and u'_i by $h = (\theta, y)$ defines the adjoint action of the Poincaré group on the basis vectors.

We can now construct geometric actions which are invariant under Poincaré transformations. For this, we let the action depend only on the components V and W of $g^{-1}dg$. It

will then automatically be invariant under (left) Poincaré transformations (1). With this prescription, we will arrive at a general description for relativistic objects with spin. To illustrate this we first review the case of relativistic particles.[1]

2.2 Spinning Particle Action

The particle action is constructed in terms of the pull-back of $g^{-1}dg$ to the world line. The most general geometric particle action which we can write in this way is

$$S = S_k(\Lambda, x) = S_1 + S_2 , \quad (8)$$

where

$$S_1 = \int k_i^{(1)} V^i , \quad (9)$$

$$S_2 = -\frac{1}{2} \int k_{ij}^{(2)} W^{ij} , \quad (10)$$

and $k_i^{(1)}$ and $k_{ij}^{(2)}$ are constants, the latter being antisymmetric in the indices i and j . In d space-time dimensions there are then a total of $\frac{1}{2}d(d+1)$ constants. These constants determine the particle dynamics. Actually for that purpose, we only need to specify certain ‘orbits’ of $k_i^{(1)}$ and $k_{ij}^{(2)}$. These orbits are induced by the action of the Poincaré group. We define this action by the following set of transformations $k \rightarrow k'$:

$$\begin{aligned} k_i^{(1)} &\rightarrow k'_i{}^{(1)} = [^\theta k^{(1)}]_i \\ k_{ij}^{(2)} &\rightarrow k'_{ij}{}^{(2)} = [^\theta k^{(2)}]_{ij} - [^\theta k^{(1)}]_i y_j + [^\theta k^{(1)}]_j y_i , \end{aligned} \quad (11)$$

where

$$\begin{aligned} [^\theta k^{(1)}]_i &= \theta_i{}^r k_r^{(1)} , \\ [^\theta k^{(2)}]_{ij} &= \theta_i{}^r \theta_j{}^s k_{rs}^{(2)} . \end{aligned} \quad (12)$$

$k_i^{(1)}$ and $k_{ij}^{(2)}$ thus transform under the Poincaré group like momentum and angular momentum. The orbits can be classified by their invariants which are the usual ones for the Poincaré algebra. In four space-time dimensions they are $k_i^{(1)}k^{(1)i}$ and $\omega_i\omega^i$, where $\omega^i = \epsilon^{ijkl}k_j^{(1)}k_{kl}^{(2)}$.

Using (11) it can be shown that

$$S_{k'}(\Lambda, x) = S_k(\Lambda\theta, \Lambda y + x) . \quad (13)$$

Thus, replacing $k_i^{(1)}$ and $k_{ij}^{(2)}$ in the action by $k'_i{}^{(1)}$ and $k'_{ij}{}^{(2)}$ is equivalent to transforming the variables Λ and x by the right action (2) of the Poincaré group. The equations of motion obtained by varying $\Lambda' = \Lambda\theta$ and $x' = \Lambda y + x$ in (13) are identical to those resulting from variations of Λ and x in (8). Therefore (11) define maps between equivalent dynamical systems.

The term S_1 in the action (8) by itself describes a spinless particle, while S_2 (which is a Wess-Zumino term for this system) gives rise to spin. This is easily seen from the equations of motion, which we can obtain by extremizing the action with respect to Poincaré transformations. The equations of motion state that there are two sets of constants of the motion. From infinitesimal translations, $\delta x^i = \epsilon^i$ and $\delta\Lambda^i{}_j = 0$, we get the constants of motion associated with the momentum,

$$p_i = [{}^\Lambda k^{(1)}]_i . \quad (14)$$

From infinitesimal Lorentz transformations, $\delta x^i = \rho^i{}_j x^j$, $\delta\Lambda^i{}_j = \rho^i{}_k \Lambda^k{}_j$, for infinitesimal $\rho_{ij} = -\rho_{ji}$, we get constants of motion associated with angular momentum,

$$j_{ij} = x_i p_j - x_j p_i + [{}^\Lambda k^{(2)}]_{ij} . \quad (15)$$

$s_{ij} = [{}^\Lambda k^{(2)}]_{ij}$ then gives the spin contribution of the particle to the total angular momentum.

As usual, p_i and j_{ij} are not all independent. For example, for a massive spinning particle they are subject to two conditions:

$$p^i p_i = k_i^{(1)} k^{(1)i} \quad \text{and} \quad w^i w_i = \omega_i \omega^i , \quad (16)$$

where w^i is the Pauli-Lubanski vector $w^i = \epsilon^{ijk\ell} p_j j_{k\ell}$. The quantum analogues of p_i and j_{ij} generate the Poincaré algebra in the quantum theory. Their representations must be irreducible, the particular representation being determined by the orbit of k via conditions like (16) on the quantum operators.

2.3 Spinning String Action

We now adapt a similar approach to the description of spinning strings.

A geometric action for strings can be expressed as a wedge products of the one forms V and W defined in eq. (5). There are thus three possible terms:

$$S = S_K(\Lambda, x) = S_1 + S_2 + S_3 , \quad (17)$$

where

$$S_1 = \frac{1}{2} \int K_{ij}^{(1)} V^i \wedge V^j , \quad (18)$$

$$S_2 = \int K_{ijk}^{(2)} V^i \wedge W^{jk} , \quad (19)$$

$$S_3 = \frac{1}{8} \int K_{ijkl}^{(3)} W^{ij} \wedge W^{kl} . \quad (20)$$

$K = (K_{ij}^{(1)}, K_{ijk}^{(2)}, K_{ijkl}^{(3)})$ denotes a set of constants and they are the analogues of the constants $k_i^{(1)}$ and $k_{ij}^{(2)}$ appearing in the particle action. They satisfy the following symmetry properties:

$$K_{ij}^{(1)} = -K_{ji}^{(1)} ,$$

$$\begin{aligned} K_{ijk}^{(2)} &= -K_{ikj}^{(2)}, \\ K_{ijk\ell}^{(3)} &= -K_{jik\ell}^{(3)} = -K_{k\ell ij}^{(3)} = -K_{ij\ell k}^{(3)}. \end{aligned} \quad (21)$$

In d space-time dimensions there are then a total of $\frac{1}{8}d(d-1)(d+1)(d+2)$ constants K .

The constants K determine the string dynamics. In analogy with the particle description, we only need to specify the ‘orbits’ on which K lie. These orbits are again induced by the action of the Poincaré group. We define this action by the following set of transformations $K \rightarrow K'$:

$$\begin{aligned} K_{ij}^{(1)} &\rightarrow K'_{ij}^{(1)} = [{}^\theta K^{(1)}]_{ij} \\ K_{ijk}^{(2)} &\rightarrow K'_{ijk}^{(2)} = [{}^\theta K^{(2)}]_{ijk} + \frac{1}{2}[{}^\theta K^{(1)}]_{ij} y_k - \frac{1}{2}[{}^\theta K^{(1)}]_{ik} y_j, \\ K_{ijkl}^{(3)} &\rightarrow K'_{ijkl}^{(3)} = [{}^\theta K^{(3)}]_{ijkl} + 2[{}^\theta K^{(2)}]_{ikl} y_j + 2[{}^\theta K^{(2)}]_{\ell ij} y_k - 2[{}^\theta K^{(2)}]_{kij} y_\ell - 2[{}^\theta K^{(2)}]_{jkl} y_i \\ &\quad + [{}^\theta K^{(1)}]_{ik} y_j y_\ell - [{}^\theta K^{(1)}]_{i\ell} y_j y_k - [{}^\theta K^{(1)}]_{jk} y_i y_\ell + [{}^\theta K^{(1)}]_{j\ell} y_i y_k, \end{aligned} \quad (22)$$

where

$$\begin{aligned} [{}^\theta K^{(1)}]_{ij} &= \theta_i^r \theta_j^s K_{rs}^{(1)}, \\ [{}^\theta K^{(2)}]_{ijk} &= \theta_i^r \theta_j^s \theta_k^t K_{rst}^{(2)}, \\ [{}^\theta K^{(3)}]_{ijkl} &= \theta_i^r \theta_j^s \theta_k^t \theta_\ell^u K_{rstu}^{(3)}. \end{aligned} \quad (23)$$

Using these definitions it can be shown that

$$S_{K'}(\Lambda, x) = S_K(\Lambda\theta, \Lambda y + x), \quad (24)$$

in analogy to (13). Thus, replacing K in the action by K' is equivalent to transforming the variables Λ and x by the right action (2) of the Poincaré group. The equations of motion

obtained by varying $\Lambda' = \Lambda\theta$ and $x' = \Lambda y + x$ in (24) are identical to those resulting from variations of Λ and x in (17). Therefore (22) define maps between equivalent dynamical systems.

The orbits induced by the action (22) of the Poincaré group on K can be labeled by their invariants. One simple quadratic invariant is of course

$$\mathrm{Tr}K^{(1)^2} .$$

In general, the expression for the invariant depends on the number of space-time dimensions. For the example of three space-time dimensions, we found the following additional quadratic invariant: [2]

$$\epsilon^{ijk}\epsilon^{\ell mn}(K_{\ell jk}^{(2)}K_{imn}^{(2)} - \frac{1}{4}K_{ijk\ell}^{(3)}K_{mn}^{(1)}) .$$

In four space-time dimensions, one instead has the quadratic invariant:[4]

$$\epsilon^{ijk\ell}K_{ij}^{(1)}K_{k\ell}^{(1)} .$$

3 Equations of Motion

We next examine the classical string dynamics following from the actions i) S_1 , ii) S_2 , iii) S_3 and iv) $S = S_1 + S_2 + S_3$.

i) The action S_1 , which can be expressed by

$$S_1 = \frac{1}{2}L_{ij}^{(1)} dx^i \wedge dx^j , \quad (25)$$

was discussed in ref. [4] and for certain orbits of K is known to be equivalent to the Nambu action. Here we define $L_{ij}^{(1)} = [\Lambda K^{(1)}]_{ij} = \Lambda_i{}^r \Lambda_j{}^s K_{rs}^{(1)}$. The standard form of the Nambu action is obtained upon eliminating $\Lambda^i{}_j$ (which in this case play the role of auxiliary

variables) from S_1 . For other orbits of K , the action S_1 can yield either tachyonic or null strings[5].

For all choices of $K_{ij}^{(1)}$ the action S_1 alone describes a spinless string. This will be evident from the form of the conserved momenta and angular momenta. These conserved currents are found by extremizing the action with respect to Poincaré transformations. From infinitesimal translations $\delta x^i = \epsilon^i$ and $\delta \Lambda^i_j = 0$, we get the equations of motion corresponding to momentum current conservation,

$$\partial_\alpha P_{(1)i}^\alpha = 0 , \quad P_{(1)i}^\alpha = \epsilon^{\alpha\beta} L_{ij}^{(1)} \partial_\beta x^j , \quad (26)$$

where $\alpha, \beta, ..$ denote world sheet indices and $\epsilon^{01} = -\epsilon^{10} = 1$. From infinitesimal Lorentz transformations, $\delta x^i = \rho^i_j x^j$, $\delta \Lambda^i_j = \rho^i_k \Lambda^k_j$, for infinitesimal $\rho_{ij} = -\rho_{ji}$, we get angular momentum current conservation,

$$\partial_\alpha J_{(1)ij}^\alpha = 0 , \quad J_{(1)ij}^\alpha = x_i P_{(1)j}^\alpha - x_j P_{(1)i}^\alpha . \quad (27)$$

Consequently, the angular momentum current consists of only an orbital term, and therefore the string defined by the action S_1 is spinless.

ii) A spin contribution to the angular momentum current is present for strings described by the action S_2 , which can also be written as

$$S_2 = L_{ijk}^{(2)} dx^i \wedge (d\Lambda \Lambda^{-1})^{jk} , \quad (28)$$

where $L_{ijk}^{(2)} = [{}^A K^{(2)}]_{ijk} = \Lambda_i^r \Lambda_j^s \Lambda_k^t K_{rst}^{(2)}$. Now from infinitesimal translations $\delta x^i = \epsilon^i$ and $\delta \Lambda^i_j = 0$, we get the equations of motion

$$\partial_\alpha P_{(2)i}^\alpha = 0 , \quad P_{(2)i}^\alpha = \epsilon^{\alpha\beta} L_{ijk}^{(2)} (\partial_\beta \Lambda \Lambda^{-1})^{jk} . \quad (29)$$

From infinitesimal Lorentz transformations, $\delta x^i = \rho^i_j x^j$, $\delta \Lambda^i_j = \rho^i_k \Lambda^k_j$, we get that

$$\partial_\alpha J_{(2)ij}^\alpha = 0, \quad J_{(2)ij}^\alpha = x_i P_{(2)j}^\alpha - x_j P_{(2)i}^\alpha + S_{(2)ij}^\alpha. \quad (30)$$

$S_{(2)ij}^\alpha$ denotes the spin contribution to the angular momentum current. It is given by

$$S_{(2)ij}^\alpha = 2\epsilon^{\alpha\beta} L_{kij}^{(2)} \partial_\beta x^k. \quad (31)$$

iii) We next consider the case of strings described by the action S_3 , which can be written

$$S_3 = \frac{1}{8} L_{ijkl}^{(3)} (d\Lambda \Lambda^{-1})^{ij} \wedge (d\Lambda \Lambda^{-1})^{kl}, \quad (32)$$

where $L_{ijkl}^{(3)} = [{}^\Lambda K^{(3)}]_{ijkl} = \Lambda_i^r \Lambda_j^s \Lambda_k^t \Lambda_\ell^u K_{rstu}^{(3)}$. Now the angular momentum current consists solely of a spin term. This is evident because the momentum current vanishes, $P_{(3)i}^\alpha = 0$, and hence so does the orbital angular momentum. Here the spin current has the following form:

$$J_{(3)ij}^\alpha = S_{(3)ij}^\alpha = -\frac{1}{2} \epsilon^{\alpha\beta} L_{ijkl}^{(3)} (\partial_\beta \Lambda \Lambda^{-1})^{kl}, \quad (33)$$

which is seen by extremizing S_3 with respect to variations $\delta \Lambda^i_j = \rho^i_k \Lambda^k_j$. In 2+1 space-time dimensions, S_3 was seen to be the integral of an exact two form, and as a result there $J_{(3)ij}^\alpha$ was identically conserved.[2]

iv) For the most general action (17) consisting of all three terms S_1 , S_2 and S_3 , the conserved momentum and angular momentum currents are given by the sum of the individual currents:

$$P_i^\alpha = P_{(1)i}^\alpha + P_{(2)i}^\alpha = \epsilon^{\alpha\beta} \left(L_{ij}^{(1)} \partial_\beta x^j + L_{ijk}^{(2)} (\partial_\beta \Lambda \Lambda^{-1})^{jk} \right), \quad (34)$$

$$J_{ij}^\alpha = J_{(1)ij}^\alpha + J_{(2)ij}^\alpha + J_{(3)ij}^\alpha. \quad (35)$$

The angular momentum current can be expressed as a sum of an orbital part $x_i P_j^\alpha - x_j P_i^\alpha$ and a spin current. The latter is given by

$$S_{ij}^\alpha = S_{(2)ij}^\alpha + S_{(3)ij}^\alpha = \epsilon^{\alpha\beta} \left(2 L_{kij}^{(2)} \partial_\beta x^k - \frac{1}{2} L_{ijkl}^{(3)} (\partial_\beta \Lambda \Lambda^{-1})^{kl} \right). \quad (36)$$

We note that the tensors $L_{ij}^{(1)}$, $L_{ijk}^{(2)}$ and $L_{ijkl}^{(3)}$ which appear in the string equations of motion satisfy the same symmetry properties as the constant tensors $K_{ij}^{(1)}$, $K_{ijk}^{(2)}$ and $K_{ijkl}^{(3)}$, i.e., we can replace K by L in (21).

4 Constraint Analysis and Current Algebra

We next proceed with the Hamiltonian formulation of the system. Constraints are present in the Hamiltonian description since all terms in the action (17) are first order in world sheet time (τ -) derivatives. Furthermore, the constraints can be either first or second class. As a result of this, the analysis of the current algebra is quite involved.

Below we shall handle the cases i)-iv) separately. Before doing so, however, we shall guess the expression for the diffeomorphism generators on a fixed time slice of the string world sheet, and show that it satisfies the Virasoro algebra.

We first introduce the momentum variables π_i and Σ_{ij} , which along with x^i and Λ span the phase space. The variables π_i are defined to be canonically conjugate to x^i , while Σ_{ij} generate left transformations on Λ . This can be expressed in terms of the following equal (world sheet) time Poisson brackets:

$$\begin{aligned} \{x_i(\sigma), \pi_j(\sigma')\} &= \eta_{ij} \delta(\sigma - \sigma') , \\ \{\Lambda(\sigma), \Sigma_{ij}(\sigma')\} &= [t_{ij}\Lambda](\sigma) \delta(\sigma - \sigma') , \end{aligned}$$

$$\{\Sigma_{ij}(\sigma), \Sigma_{k\ell}(\sigma')\} = [\eta_{ik}\Sigma_{j\ell} + \eta_{jk}\Sigma_{\ell i} + \eta_{i\ell}\Sigma_{kj} + \eta_{j\ell}\Sigma_{ik}](\sigma) \delta(\sigma - \sigma') , \quad (37)$$

where we write the phase space variables on a world sheet time slice, the phase space variable being periodic functions of the spatial coordinate σ of the world sheet. All other Poisson brackets between the phase space variables are zero. Below we shall utilize the following matrix representation for t_{ij} :

$$(t_{ij})_{k\ell} = -\eta_{ik}\eta_{j\ell} + \eta_{i\ell}\eta_{jk} . \quad (38)$$

As mentioned above, the constraints can be classified as both first and second class. In this regard, we know from the reparametrization symmetry of the action, that there exist some combinations of the constraints which are first class. Those which generate diffeomorphisms along a $\tau = constant$ surface should satisfy the Virasoro algebra. It is easy to construct such generators. They are:

$$\mathcal{L}[f] = \int d\sigma f(\sigma) \left(\pi_i \partial_\sigma x^i - \frac{1}{2} \Sigma_{ij} (\partial_\sigma \Lambda \Lambda^{-1})^{ij} \right) , \quad (39)$$

where f are periodic functions on a fixed τ slice. From (37), it can be verified that $\mathcal{L}[f]$ indeed do satisfy the Virasoro algebra,

$$\{\mathcal{L}[f], \mathcal{L}[f']\} = \mathcal{L}[f \partial_\sigma f' - f' \partial_\sigma f] . \quad (40)$$

In the subsections that follow, we shall show that $\mathcal{L}[f]$ are first class constraints. For this we shall need the specific form for the action in order to obtain the explicit expressions for the constraints, which look different in the four different cases mentioned earlier. With this in mind we now specialize to the cases i)-iv).

4.1 Case i)

As stated before, case i) describes a spinless string. The constraints on the momentum variables are:

$$\begin{aligned}\psi_i &= \pi_i - L_{ij}^{(1)} \partial_\sigma x^j \approx 0 , \\ \phi_{ij} &= \Sigma_{ij} \approx 0 .\end{aligned}\tag{41}$$

From the first constraint and (26), π_i is thus identified with the time component of the momentum current $P_{(1)i}^0$. In addition to π_i , we can define the variables

$$\mathcal{J}_{ij} = x_i \pi_j - x_j \pi_i + \Sigma_{ij} ,\tag{42}$$

which are weakly equal (i.e., up to a linear combination of constraints) to the time component of the angular momentum current $J_{(1)ij}^0$.

To compute their Poisson brackets, we find it convenient to write the constraints as distributions:

$$\begin{aligned}\Psi[\lambda] &= \int d\sigma \lambda^i(\sigma) \psi_i(\sigma) \approx 0 , \\ \Phi[\rho] &= \int d\sigma \rho^{ij}(\sigma) \phi_{ij}(\sigma) \approx 0 ,\end{aligned}\tag{43}$$

where λ^i and $\rho^{ij} = -\rho^{ji}$ are periodic functions of the spatial coordinate σ . From the Poisson brackets (37), we obtain the following algebra of the constraints:

$$\begin{aligned}\{\Psi[\lambda], \Psi[\lambda']\} &= \int d\sigma (\partial_\sigma \lambda^j \lambda'^i - \partial_\sigma \lambda'^j \lambda^i) L_{ij}^{(1)} , \\ \{\Psi[\lambda], \Phi[\rho]\} &\approx 2 \int d\sigma \rho^{ij} (\lambda_i \pi_j + \lambda^k \partial_\sigma x_i L_{kj}^{(1)}) , \\ \{\Phi[\rho], \Phi[\rho']\} &\approx 0 .\end{aligned}\tag{44}$$

Note that the last two relations only hold weakly.

From the reparametrization symmetry of the action, we know that there exist linear combinations of $\Psi[\lambda]$ and $\Phi[\rho]$ which are first class constraints. Let

$$\Theta[\hat{\lambda}, \hat{\rho}] = \Psi[\hat{\lambda}] + \Phi[\hat{\rho}] \quad (45)$$

be a general first class constraint, where the Lagrange multipliers $\hat{\lambda}$ and $\hat{\rho}$ denote the functions $\hat{\lambda}^i(\sigma)$ and $\hat{\rho}^{ij}(\sigma) = -\hat{\rho}^{ji}(\sigma)$, respectively. They are solutions to the following equations:

$$\begin{aligned} \hat{\rho}_i^j \pi_j &\approx \frac{1}{2} \hat{\lambda}^j \partial_\sigma L_{ji}^{(1)} + \hat{\rho}^{jk} \partial_\sigma x_k L_{ij}^{(1)}, \\ \hat{\lambda}_i \pi_j - \hat{\lambda}_j \pi_i &\approx \hat{\lambda}^k (L_{jk}^{(1)} \partial_\sigma x_i - L_{ik}^{(1)} \partial_\sigma x_j), \end{aligned} \quad (46)$$

which follow from demanding that Θ is first class. Since these equations are linear in $\hat{\lambda}$ and $\hat{\rho}$, it follows that if $\hat{\lambda}$ and $\hat{\rho}$ are solutions to (46), then $f\hat{\lambda}$ and $f\hat{\rho}$ are also solutions to (46), where f is an arbitrary function on the world sheet. The generators of diffeomorphisms of the world sheet then have the general form: $\Theta[f\hat{\lambda}, f\hat{\rho}]$.

One solution to equations (46) is $\hat{\lambda} = \partial_\sigma x$ and $\hat{\rho} = -\frac{1}{2}\partial_\sigma \Lambda \Lambda^{-1}$. The resulting first class constraint (corresponding to $f = 1$),

$$H_\sigma = \Theta[\hat{\lambda}, \hat{\rho}] = \Psi[\partial_\sigma x] + \Phi\left[-\frac{1}{2}\partial_\sigma \Lambda \Lambda^{-1}\right] = \int d\sigma \left(\pi_i \partial_\sigma x^i - \frac{1}{2} \Sigma_{ij} (\partial_\sigma \Lambda \Lambda^{-1})^{ij} \right), \quad (47)$$

generates translations in σ :

$$\begin{aligned} \partial_\sigma x^i(\sigma) &= \{x^i(\sigma), H_\sigma\}, \\ \partial_\sigma \Lambda^i{}_j(\sigma) &= \{\Lambda^i{}_j(\sigma), H_\sigma\}. \end{aligned} \quad (48)$$

When $f \neq 1$ we get the generators of diffeomorphisms

$$\Theta[f\hat{\lambda}, f\hat{\rho}] = \mathcal{L}[f], \quad (49)$$

on a fixed τ slice of the world sheet. $\mathcal{L}[f]$ were given in (39) and satisfy the Virasoro algebra (40). We have therefore shown that the Virasoro generators are first class constraints.

For the case where $\Theta[\hat{\lambda}, \hat{\rho}] = H_\tau$ is the generator of τ translations, we need that $\hat{\lambda}_i$ is a time-like vector. By computing the Hamilton equations of motion for $\pi_i(\sigma)$ and $\mathcal{J}_{ij}(\sigma)$, we can recover the current conservation law for momentum and angular momentum. This follows because the Poisson brackets of $\pi_i(\sigma)$ and $\mathcal{J}_{ij}(\sigma)$ with the constraints are spatial derivatives,

$$\begin{aligned}\{\Psi[\lambda], \pi_i(\sigma)\} &= \partial_\sigma(\lambda^k L_{ki}^{(1)}) , \\ \{\Phi[\rho], \pi_i(\sigma)\} &= 0 , \\ \{\Psi[\lambda], \mathcal{J}_{ij}(\sigma)\} &\approx \partial_\sigma\left(\lambda^k(x_i L_{kj}^{(1)} - x_j L_{ki}^{(1)})\right) , \\ \{\Phi[\rho], \mathcal{J}_{ij}(\sigma)\} &\approx 0 .\end{aligned}\tag{50}$$

The Hamilton equations are then:

$$\begin{aligned}\partial_\tau \pi_i(\sigma) &= \{\pi_i(\sigma), H_\tau\} = -\partial_\sigma(\hat{\lambda}^k L_{ki}^{(1)}) , \\ \partial_\tau \mathcal{J}_{ij}(\sigma) &= \{\mathcal{J}_{ij}(\sigma), H_\tau\} \approx -\partial_\sigma\left(2\hat{\lambda}^k(x_i L_{kj}^{(1)} - x_j L_{ki}^{(1)})\right) .\end{aligned}\tag{51}$$

Here we get the identification of $\hat{\lambda}^k L_{ki}^{(1)}$ with the space component of the momentum current $P_{(1)i}^1$ defined in (26). These equations once again show that the case i) string is spinless.

The generators of the Poincaré symmetry are the charges

$$\int d\sigma \pi_i \quad \text{and} \quad \int d\sigma \mathcal{J}_{ij} .\tag{52}$$

They have zero Poisson brackets with the constraints. This again follows because the Poisson brackets of $\pi_i(\sigma)$ and $\mathcal{J}_{ij}(\sigma)$ with the constraints are spatial derivatives. The charges (52) are thus first class variables as well as Dirac variables.

It remains to construct the Dirac variables associated with current densities $\pi_i(\sigma)$ and $\mathcal{J}_{ij}(\sigma)$. For this we first define

$$\tilde{\pi}[\gamma] = \int d\sigma \gamma^i(\sigma) \pi_i(\sigma) + \Phi[E_{(\gamma)}] . \quad (53)$$

$E_{(\gamma)}$ denote the functions $E_{(\gamma)}^{ij}(\sigma) = -E_{(\gamma)}^{ji}(\sigma)$. $\tilde{\pi}[\gamma]$ have (weakly) zero Poisson brackets with $\Phi[\rho]$. For them to have zero Poisson brackets with $\Psi[\lambda]$, we need that the functions $E_{(\gamma)}^{ij}(\sigma)$ satisfy:

$$-\frac{1}{2}\partial_\sigma \gamma^i L_{ki}^{(1)} + E_{(\gamma)kj} \pi^j + E_{(\gamma)}^{ij} L_{kj}^{(1)} \partial_\sigma x_i \approx 0 . \quad (54)$$

Similarly, we can define the variables

$$\tilde{\mathcal{J}}[\xi] = \int d\sigma \xi^{ij}(\sigma) \mathcal{J}_{ij}(\sigma) + \Phi[G_{(\xi)}] . \quad (55)$$

$G_{(\xi)}$ denote the functions $G_{(\xi)}^{ij}(\sigma) = -G_{(\xi)}^{ji}(\sigma)$. Like $\tilde{\pi}[\gamma]$, they have (weakly) zero Poisson brackets with $\Phi[\rho]$. For them to have zero Poisson brackets with $\Psi[\lambda]$, we need that $G_{(\gamma)}^{ij}(\sigma)$ satisfy:

$$G_{(\xi)ki} \pi^i + G_{(\xi)}^{ij} \partial_\sigma x_i L_{kj}^{(1)} - \partial_\sigma \xi^{ij} x_i L_{kj}^{(1)} \approx 0 . \quad (56)$$

$\tilde{\pi}[\gamma]$ and $\tilde{\mathcal{J}}[\xi]$ are then Dirac variables. Since they have zero Poisson brackets with first, as well as second class constraints, they are gauge invariant.

We next compute the Poisson bracket (or current) algebra for the momenta and angular momenta. For this purpose it turns out, in this case, not to be necessary to solve (56) for $G_{(\xi)ki}$. The current algebra is simply the algebra of the Poincaré loop group:

$$\begin{aligned} \{\tilde{\pi}[\gamma], \tilde{\pi}[\gamma']\} &= 0 , \\ \{\tilde{\mathcal{J}}[\xi], \tilde{\pi}[\gamma]\} &\approx -2\tilde{\pi}[\xi\gamma] , \\ \{\tilde{\mathcal{J}}[\xi], \tilde{\mathcal{J}}[\xi']\} &\approx -4\tilde{\mathcal{J}}[\xi\xi'] . \end{aligned} \quad (57)$$

$\tilde{\mathcal{J}}[\xi]$ and $\tilde{\pi}[\gamma]$ are gauge invariant coordinates which label the reduced phase space. Whether or not they form a complete set of variables is not evident. On the other hand, we note that for spinless strings $\tilde{\mathcal{J}}[\xi]$ and $\tilde{\pi}[\gamma]$ are not independent on the reduced phase space, since from (41) and (42), are \mathcal{J}_{ij} and π_k are subject to:

$$\mathcal{J}_{ij}\pi_k + \mathcal{J}_{jk}\pi_i + \mathcal{J}_{ki}\pi_j \approx 0 . \quad (58)$$

4.2 Case ii)

This case is of interest because it yields a nontrivial momentum and angular momentum current, the latter containing a spin contribution $S_{(2)ij}^\alpha$. Here we shall derive a current algebra which is an extension of the Poincaré loop group.

In this case the constraints on the momentum variables are:

$$\begin{aligned} \psi_i &= \pi_i - L_{ijk}^{(2)}(\partial_\sigma \Lambda \Lambda^{-1})^{jk} \approx 0 , \\ \phi_{ij} &= \Sigma_{ij} - 2L_{kij}^{(2)}\partial_\sigma x^k \approx 0 . \end{aligned} \quad (59)$$

From the first constraint and (29), π_i is thus identified with the time component of the momentum current $P_{(2)i}^0$, while from the second constraint and (31) Σ_{ij} is identified with the time component of the spin current $S_{(2)ij}^0$. In addition to π_i and Σ_{ij} , we can once again define the variables \mathcal{J}_{ij} , as was done in (42), which are weakly equal to the angular momentum densities $J_{(2)ij}^0$.

For the algebra of the constraints we now get:

$$\begin{aligned} \{\Psi[\lambda], \Psi[\lambda']\} &= 0 , \\ \{\Psi[\lambda], \Phi[\rho]\} &\approx 2 \int d\sigma \rho^{ij} (\lambda_i \pi_j - \lambda^k \partial_\sigma L_{kij}^{(2)}) , \end{aligned}$$

$$\{\Phi[\rho], \Phi[\rho']\} \approx 4 \int d\sigma \left((\rho^{ij} \rho'^{k\ell} - \rho'^{ij} \rho^{k\ell}) L_{\ell ij}^{(2)} \partial_\sigma x_k - \rho'^i{}_k \rho^{kj} \Sigma_{ij} \right), \quad (60)$$

where $\Psi[\lambda]$ and $\Phi[\rho]$ are once again the distributions defined in (43). As with case i), from the reparametrization symmetry of the action, we know that there exist linear combinations of $\Psi[\lambda]$ and $\Phi[\rho]$ which are first class constraints. First class constraints may also arise due to additional symmetries associated with some particular choices for $K^{(2)}$. For $\Theta[\hat{\lambda}, \hat{\rho}]$ as defined in (45) to be a first class constraint, we need that the Lagrange multipliers $\hat{\lambda}$ and $\hat{\rho}$ now satisfy:

$$\begin{aligned} \hat{\rho}_i^j \pi_j &\approx \hat{\rho}^{jk} \partial_\sigma L_{ijk}^{(2)}, \\ \frac{1}{2}(\hat{\lambda}_i \pi_j - \hat{\lambda}_j \pi_i) - \hat{\rho}_i^k \Sigma_{kj} + \hat{\rho}_j^k \Sigma_{ki} \\ &\approx \hat{\lambda}^k \partial_\sigma L_{kij}^{(2)} - \hat{\rho}^{k\ell} (L_{jk\ell}^{(2)} \partial_\sigma x_i - L_{ik\ell}^{(2)} \partial_\sigma x_j + 2L_{kij}^{(2)} \partial_\sigma x_\ell). \end{aligned} \quad (61)$$

Since, like (46), these equations are linear in $\hat{\lambda}$ and $\hat{\rho}$, it again follows that if $\hat{\lambda}$ and $\hat{\rho}$ are also solutions to (61), then so are $f\hat{\lambda}$ and $f\hat{\rho}$, where f is an arbitrary function on the world sheet. The generators of diffeomorphisms of the world sheet will again have the general form: $\Theta[f\hat{\lambda}, f\hat{\rho}]$. $\hat{\lambda} = \partial_\sigma x$ and $\hat{\rho} = -\frac{1}{2}\partial_\sigma \Lambda \Lambda^{-1}$ are once again solutions to the above conditions, with the resulting first class constraint (corresponding to $f = 1$) being equal to H_σ in (47). As before, H_σ generates translations on a $\tau = \text{constant}$ slice of the world sheet (48). When $f \neq 1$ we recover the generators of diffeomorphisms $\mathcal{L}[f] = \Theta[f\hat{\lambda}, f\hat{\rho}]$ on a fixed τ slice of the world sheet, which satisfy the Virasoro algebra (40). Thus we have again shown that the Virasoro generators are first class constraints.

For the case where $\Theta[\hat{\lambda}, \hat{\rho}] = H_\tau$ is the generator of τ translations, we again need that $\hat{\lambda}_i$ is a time-like vector. To obtain the Hamilton equations of motion for $\pi_i(\sigma)$ and $\mathcal{J}_{ij}(\sigma)$, we

can use

$$\begin{aligned}
\{\Psi[\lambda], \pi_i(\sigma)\} &= 0, \\
\{\Phi[\rho], \pi_i(\sigma)\} &= 2 \partial_\sigma (L_{ijk}^{(2)} \rho^{jk}), \\
\{\Psi[\lambda], \mathcal{J}_{ij}(\sigma)\} &\approx -2 \partial_\sigma (\lambda^k L_{kij}^{(2)}), \\
\{\Phi[\rho], \mathcal{J}_{ij}(\sigma)\} &\approx -2 \partial_\sigma \left(\rho^{kl} (L_{ikl}^{(2)} x_j - L_{jkl}^{(2)} x_i) \right).
\end{aligned} \tag{62}$$

Thus as in the previous case, the Poisson brackets of $\pi_i(\sigma)$ and $\mathcal{J}_{ij}(\sigma)$ with the constraints are spatial derivatives. The current conservation law for momentum and angular momentum then follows:

$$\begin{aligned}
\partial_\tau \pi_i(\sigma) &= \{\pi_i(\sigma), H_\tau\} = -\partial_\sigma (2L_{ijk} \hat{\rho}^{jk}), \\
\partial_\tau \mathcal{J}_{ij}(\sigma) &= \{\mathcal{J}_{ij}(\sigma), H_\tau\} \approx -\partial_\sigma \left(2\hat{\rho}^{kl} (x_i L_{jkl} - x_j L_{ikl}) - 2\hat{\lambda}^k L_{kij} \right).
\end{aligned} \tag{63}$$

Here we get the identification of $2L_{ijk}^{(2)} \hat{\rho}^{jk}$ with the space component of the momentum current $P_{(2)i}^1$, defined in (29), and $-2\hat{\lambda}^k L_{kij}$ with the space component of the spin current $S_{(2)ij}^1$, defined in (31).

As before, the charges (52) are the generators of the Poincaré symmetry. From (62), they have zero Poisson brackets with the constraints, and therefore they are Dirac variables. We can construct the Dirac variables associated with current densities $\pi_i(\sigma)$ and $\mathcal{J}_{ij}(\sigma)$, in a similar manner to what was done previously. We define

$$\tilde{\pi}[\gamma] = \int d\sigma \gamma^i(\sigma) \pi_i(\sigma) + \Psi[F_{(\gamma)}]. \tag{64}$$

$F_{(\gamma)}$ denote the functions $F_{(\gamma)}^j(\sigma)$. From the first equations in (60) and (62), $\tilde{\pi}[\gamma]$ have zero Poisson brackets with $\Psi[\lambda]$. For them to have zero Poisson brackets with $\Phi[\rho]$, we need that the functions $F_{(\gamma)}^j(\sigma)$ satisfy:

$$\partial_\sigma \gamma^i L_{ijk}^{(2)} + \frac{1}{2} (F_{(\gamma)j} \pi_k - F_{(\gamma)k} \pi_j) - F_{(\gamma)}^i \partial_\sigma L_{ijk}^{(2)} \approx 0. \tag{65}$$

We next define the variables

$$\tilde{\mathcal{J}}[\xi] = \int d\sigma \xi^{ij}(\sigma) \mathcal{J}_{ij}(\sigma) + \Psi[H_{(\xi)}] + \Phi[G_{(\xi)}] . \quad (66)$$

$H_{(\xi)}$ and $G_{(\xi)}$ denote the functions $H_{(\xi)}^i(\sigma)$ and $G_{(\xi)}^{ij}(\sigma) = -G_{(\xi)}^{ji}(\sigma)$, respectively. For $\tilde{\mathcal{J}}[\xi]$ to have zero Poisson brackets with all of the constraints, we need that $H_{(\xi)}$ and $G_{(\xi)}$ satisfy:

$$\begin{aligned} \partial_\sigma \xi^{ij} L_{kij}^{(2)} + G_{(\xi)ki} \pi^i - G_{(\xi)}^{ij} \partial_\sigma L_{kij}^{(2)} &\approx 0 , \\ 2\partial_\sigma \xi^{ij} L_{ik\ell}^{(2)} x_j - \frac{1}{2}(H_{(\xi)k} \pi_\ell - H_{(\xi)\ell} \pi_k) + H_{(\xi)}^i \partial_\sigma L_{ik\ell}^{(2)} \\ - G_{(\xi)}^{ij} (L_{\ell ij}^{(2)} \partial_\sigma x_k - L_{kij}^{(2)} \partial_\sigma x_\ell - 2L_{jk\ell}^{(2)} \partial_\sigma x_i) - G_{(\xi)ik} \Sigma_\ell^i + G_{(\xi)i\ell} \Sigma_k^i &\approx 0 . \end{aligned} \quad (67)$$

Due to the existence of the first class constraints Θ , $H_{(\xi)}$ and $G_{(\xi)}$ are not uniquely defined by these equations.

The current algebra generated by the momentum and angular momentum densities can be written in terms of the functions $F_{(\gamma)}$, $H_{(\xi)}$ and $G_{(\xi)}$. Unlike in the spinless case, we now find a rather complicated result:

$$\begin{aligned} \{\tilde{\pi}[\gamma], \tilde{\pi}[\gamma']\} &= 0 , \\ \{\tilde{\mathcal{J}}[\xi], \tilde{\pi}[\gamma]\} &\approx -2\tilde{\pi}[\xi\gamma] - 2\tilde{\pi}[G_{(\xi)}F_{(\gamma)}] - 2 \int d\sigma G_{(\xi)}^{ij} F_{(\gamma)}^k \partial_\sigma L_{kij}^{(2)} , \\ \{\tilde{\mathcal{J}}[\xi], \tilde{\mathcal{J}}[\xi']\} &\approx -4\tilde{\mathcal{J}}[\xi\xi'] + 2\tilde{\pi}[G_{(\xi')}H_{(\xi)}] - 2\tilde{\pi}[G_{(\xi)}H_{(\xi')}] \\ &\quad + 2 \int d\sigma \left(H_{(\xi)}^i G_{(\xi')}^{jk} \partial_\sigma L_{ijk}^{(2)} + 2G_{(\xi)}^{ij} G_{(\xi')}^{k\ell} L_{jk\ell}^{(2)} \partial_\sigma x_i + G_{(\xi)}^{ij} G_{(\xi')}^k \Sigma_{ki} - (\xi \rightleftharpoons \xi') \left. \right) \end{aligned} \quad (68)$$

We have used equations (67) to write the last Poisson bracket in a manifestly antisymmetric way.

The first terms on the right hand sides of equations (68) correspond to the algebra associated with the Poincaré loop group, while the remaining terms represent a complicated

extension of that algebra. If we make the assumption that the momentum and angular momentum currents of relativistic strings must satisfy the Poincaré loop group algebra in the quantum theory, then the above remaining terms are anomalous and they are obstructions to the quantization of the theory. Requiring that all the anomalous terms vanish in (68) yields a total of $\frac{1}{8}d(d-1)(d^2+3d-2)$ conditions on the $\frac{1}{2}d^2(d-1)$ functions $L_{ijk}^{(2)}$, and then on the $\frac{1}{2}d^2(d-1)$ constants $K_{ijk}^{(2)}$. The system looks overdetermined, however that is not entirely clear because, as stated earlier, the functions $H_{(\xi)}$ and $G_{(\xi)}$ are not uniquely defined by (67). Even if the system is overdetermined, it does not necessarily imply that there are no solutions for $K_{ijk}^{(2)}$, although we have not yet discovered any.

4.3 Case iii)

This is the case of pure spin, as the momentum current vanishes. Here the constraints are:

$$\begin{aligned}\psi_i &= \pi_i \approx 0 , \\ \phi_{ij} &= \Sigma_{ij} + \frac{1}{2}L_{ijkl}^{(3)}(\partial_\sigma \Lambda \Lambda^{-1})^{kl} \approx 0 .\end{aligned}\quad (69)$$

From the second constraint and (33), Σ_{ij} is once again identified with the time component of the spin current. Since the orbital angular momentum vanishes, Σ_{ij} is weakly equal to the total angular momentum \mathcal{J}_{ij} defined in (42).

The algebra of the constraints now is given by:

$$\begin{aligned}\{\Psi[\lambda], \Psi[\lambda']\} &= 0 , \\ \{\Psi[\lambda], \Phi[\rho]\} &= 0 , \\ \{\Phi[\rho], \Phi[\rho']\} &\approx \int d\sigma \left((\rho^{ij} \partial_\sigma \rho'^{kl} - \rho'^{ij} \partial_\sigma \rho^{kl}) L_{ijkl}^{(3)} - 4\rho^{ij} \rho'_i{}^k \Sigma_{jk} \right) ,\end{aligned}\quad (70)$$

$\Psi[\lambda]$ and $\Phi[\rho]$ again being the distributions defined in (43). It immediately follows that $\Psi[\lambda]$ for all λ are first class constraints. In order to find the remaining first class constraints we

can once again define $\Theta[\hat{\lambda}, \hat{\rho}]$ as was done in (45). Then $\hat{\rho}$ satisfies:

$$\hat{\rho}_j^k \Sigma_{ik} - \hat{\rho}_i^k \Sigma_{jk} - \frac{1}{2} \partial_\sigma L_{ijkl}^{(3)} \hat{\rho}^{kl} \approx 0 , \quad (71)$$

while $\hat{\lambda}$ is arbitrary. Since these equations are linear in $\hat{\rho}$, it follows that if $\hat{\rho}$ is a solution, then so is $f\hat{\rho}$, where f is an arbitrary function on the world sheet. Again the generators of diffeomorphisms of the world sheet can be written in the general form: $\Theta[f\hat{\lambda}, f\hat{\rho}]$, here $\hat{\lambda}$ being arbitrary, while $\hat{\rho}$ satisfies (71). A solution to the latter is $\hat{\rho} = -\frac{1}{2}\partial_\sigma \Lambda \Lambda^{-1}$. If we also fix $\hat{\lambda}$ to be equal to $\partial_\sigma x$, we recover the generators of diffeomorphisms $\mathcal{L}[f] = \Theta[f\hat{\lambda}, f\hat{\rho}]$ on a fixed τ slice of the world sheet, which satisfy the Virasoro algebra (40). Thus we have again shown that the Virasoro generators are first class constraints.

The canonical momenta π_i have zero Poisson brackets with all constraints, and hence can be eliminated from the phase space. The remaining momenta Σ_{ij} have the following Poisson brackets with the constraints $\Phi[\rho]$:

$$\{\Phi[\rho], \Sigma_{ij}(\sigma)\} \approx -\partial_\sigma (\rho^{kl} L_{ijkl}^{(3)}) . \quad (72)$$

We can choose $H_\tau = \Phi[\hat{\rho}]$ for the Hamiltonian of this system, with $\hat{\rho}$ being a solution to (71) and the identification made of $-\rho^{kl} L_{ijkl}^{(3)}$ with the space component of the spin current $S_{(3)ij}^1$ defined in (33).

From (72), the angular momentum charges

$$\int d\sigma \Sigma_{ij} \quad (73)$$

have zero Poisson brackets with the constraints and hence are Dirac variables. Once again, they then generate the Lorentz algebra. We can construct the Dirac variables associated with current densities $\Sigma_{ij}(\sigma)$, in a similar manner to what was done previously. We define

$$\tilde{\Sigma}[\xi] = \int d\sigma \xi^{ij}(\sigma) \Sigma_{ij}(\sigma) + \Phi[G_{(\xi)}] , \quad (74)$$

where as before $G_{(\xi)}$ denotes the functions $G_{(\xi)}^{ij}(\sigma) = -G_{(\xi)}^{ji}(\sigma)$. For $\tilde{\Sigma}[\xi]$ to have zero Poisson brackets with all of the constraints, we need that $G_{(\xi)}$ satisfy:

$$\partial_\sigma \xi^{k\ell} L_{ijk\ell}^{(3)} + G_{(\xi)}^{k\ell} \partial_\sigma L_{ijk\ell}^{(3)} + 2G_{(\xi)ik} \Sigma_j{}^k - 2G_{(\xi)jk} \Sigma_i{}^k \approx 0 . \quad (75)$$

Due to the existence of the first class constraints $\Phi[\hat{\rho}]$, $G_{(\xi)}$ are not uniquely defined by these equations. Now the current algebra for the angular momenta has the following form:

$$\{\tilde{\Sigma}[\xi], \tilde{\Sigma}[\xi']\} \approx -4\tilde{\Sigma}[\xi\xi'] - 4\tilde{\Sigma}[G_{(\xi)}G_{(\xi')}] + \int d\sigma G_{(\xi)}^{ij} G_{(\xi')}^{k\ell} \partial_\sigma L_{ijk\ell}^{(3)} . \quad (76)$$

This Poisson bracket satisfies the antisymmetry property.

The first term on the right hand side of (76) corresponds to the algebra associated with the Lorentz loop group, while the remaining terms again represent an extension of the algebra. If we now demand that we should get the Lorentz loop group algebra in the quantum theory, then the above remaining terms are anomalous and they are obstructions to the quantization of the theory. Requiring that all the anomalous terms vanish in (76) yields a total of $\frac{1}{8}d(d-1)(d+1)(d-2)$ conditions on the $\frac{1}{8}d(d-1)(d+1)(d-2)$ functions $L_{ijk\ell}^{(3)}$, and then on the $\frac{1}{8}d(d-1)(d+1)(d-2)$ constants $K_{ijk\ell}^{(3)}$. The system is actually underdetermined because, as stated earlier, the functions $G_{(\xi)}$ are not uniquely defined by (75). We have not yet found solutions for $K_{ijk}^{(3)}$.

4.4 Case iv)

This is the most general case of a spinning string and it is simple to put together all three of the previous cases. The constraints are now:

$$\begin{aligned} \psi_i &= \pi_i - L_{ij}^{(1)} \partial_\sigma x^j - L_{ijk}^{(2)} (\partial_\sigma \Lambda \Lambda^{-1})^{jk} \approx 0 , \\ \phi_{ij} &= \Sigma_{ij} - 2L_{kij}^{(2)} \partial_\sigma x^k + \frac{1}{2} L_{ijk\ell}^{(3)} (\partial_\sigma \Lambda \Lambda^{-1})^{k\ell} \approx 0 , \end{aligned} \quad (77)$$

and their algebra is given by:

$$\begin{aligned}
\{\Psi[\lambda], \Psi[\lambda']\} &= \int d\sigma (\partial_\sigma \lambda^j \lambda'^i - \partial_\sigma \lambda'^j \lambda^i) L_{ij}^{(1)}, \\
\{\Psi[\lambda], \Phi[\rho]\} &\approx 2 \int d\sigma \rho^{ij} \left(\lambda_i \pi_j + \lambda^k (\partial_\sigma x_i L_{kj}^{(1)} - \partial_\sigma L_{kij}^{(2)}) \right), \\
\{\Phi[\rho], \Phi[\rho']\} &\approx \int d\sigma \left(-4 \rho'^i{}_k \rho^{kj} \Sigma_{ij} + 4(\rho^{ij} \rho'^{kl} - \rho'^{ij} \rho^{kl}) L_{\ell ij}^{(2)} \partial_\sigma x_k \right. \\
&\quad \left. + (\rho^{ij} \partial_\sigma \rho'^{kl} - \rho'^{ij} \partial_\sigma \rho^{kl}) L_{ijk\ell}^{(3)} \right), \tag{78}
\end{aligned}$$

Once again we can find the subset of first class constraints for the system. They have the general form: $\Theta[f\hat{\lambda}, f\hat{\rho}]$. If we take $\hat{\rho} = -\frac{1}{2}\partial_\sigma \Lambda \Lambda^{-1}$ and $\hat{\lambda} = \partial_\sigma x$, we again recover the generators of diffeomorphisms $\mathcal{L}[f] = \Theta[f\hat{\lambda}, f\hat{\rho}]$ on a fixed τ slice of the world sheet, which satisfy the Virasoro algebra (40).

The Poisson brackets of the momentum and angular momentum current densities are again given by spatial derivatives. We get

$$\begin{aligned}
\{\Psi[\lambda], \pi_i(\sigma)\} &= \partial_\sigma (\lambda^k L_{ki}^{(1)}), \\
\{\Phi[\rho], \pi_i(\sigma)\} &= 2 \partial_\sigma (L_{ijk}^{(2)} \rho^{jk}), \\
\{\Psi[\lambda], \mathcal{J}_{ij}(\sigma)\} &\approx \partial_\sigma \left(\lambda^k (x_i L_{kj}^{(1)} - x_j L_{ki}^{(1)} - 2L_{kij}^{(2)}) \right), \\
\{\Phi[\rho], \mathcal{J}_{ij}(\sigma)\} &\approx -2 \partial_\sigma \left(\rho^{kl} (L_{ikl}^{(2)} x_j - L_{jkl}^{(2)} x_i + \frac{1}{2} L_{ijkl}^{(3)}) \right), \tag{79}
\end{aligned}$$

from which it follows that the charges (52) generating the Poincaré algebra are Dirac variables. For the remaining Dirac variables we define

$$\tilde{\pi}[\gamma] = \int d\sigma \gamma^i(\sigma) \pi_i(\sigma) + \Psi[F_{(\gamma)}] + \Phi[E_{(\gamma)}], \tag{80}$$

where $E_{(\gamma)}^{ij}(\sigma) = -E_{(\gamma)}^{ji}(\sigma)$, along with $\tilde{\mathcal{J}}[\xi]$ in (66). For $\tilde{\pi}[\gamma]$ and $\tilde{\mathcal{J}}[\xi]$ to have (weakly) zero Poisson brackets with $\Psi[\lambda]$ and $\Phi[\rho]$, we need that the functions $E_{(\gamma)}^{jk}$, $F_{(\gamma)}^k$, $H_{(\xi)}^k$ and $G_{(\xi)}^{jk}$ satisfy:

$$-\frac{1}{2} \partial_\sigma \gamma^i L_{ki}^{(1)} + E_{(\gamma)kj} \pi^j + E_{(\gamma)}^{ij} (L_{kj}^{(1)} \partial_\sigma x_i - \partial_\sigma L_{kij}^{(2)}) + \frac{1}{2} F_{(\gamma)}^j \partial_\sigma L_{kj}^{(1)} \approx 0,$$

$$\begin{aligned} & \partial_\sigma \gamma^i L_{ijk}^{(2)} + \frac{1}{2}(F_{(\gamma)j}\pi_k - F_{(\gamma)k}\pi_j) - \frac{1}{2}F_{(\gamma)}^i(L_{ij}^{(1)}\partial_\sigma x_k - L_{ik}^{(1)}\partial_\sigma x_j + 2\partial_\sigma L_{ijk}^{(2)}) \\ & + E_{(\gamma)ji}\Sigma_k{}^i - E_{(\gamma)ki}\Sigma_j{}^i - E_{(\gamma)}^{\ell m}\left(L_{j\ell m}^{(2)}\partial_\sigma x_k - L_{k\ell m}^{(2)}\partial_\sigma x_j + 2L_{mjk}^{(2)}\partial_\sigma x_\ell + \frac{1}{2}\partial_\sigma L_{\ell m j k}^{(3)}\right) \approx 0 , \end{aligned}$$

$$\partial_\sigma \xi^{ij}(L_{kij}^{(2)} - x_i L_{kj}^{(1)}) - \frac{1}{2}H_{(\xi)}^i\partial_\sigma L_{ik}^{(1)} + G_{(\xi)ki}\pi^i + G_{(\xi)}^{ij}(\partial_\sigma x_i L_{kj}^{(1)} - \partial_\sigma L_{kij}^{(2)}) \approx 0 ,$$

$$\begin{aligned} & \partial_\sigma \xi^{ij}\left(2L_{ik\ell}^{(2)}x_j + \frac{1}{2}L_{ijk\ell}^{(3)}\right) - \frac{1}{2}(H_{(\xi)k}\pi_\ell - H_{(\xi)\ell}\pi_k) + \frac{1}{2}H_{(\xi)}^i(\partial_\sigma x_\ell L_{ik}^{(1)} - \partial_\sigma x_k L_{i\ell}^{(1)} + 2\partial_\sigma L_{ik\ell}^{(2)}) \\ & - G_{(\xi)}^{ij}\left(L_{\ell ij}^{(2)}\partial_\sigma x_k - L_{kij}^{(2)}\partial_\sigma x_\ell - 2L_{jk\ell}^{(2)}\partial_\sigma x_i - \frac{1}{2}\partial_\sigma L_{ijkl}^{(3)}\right) - G_{(\xi)ik}\Sigma_\ell{}^i + G_{(\xi)i\ell}\Sigma_k{}^i \approx 0 . \quad (81) \end{aligned}$$

Due to the existence of the first class constraints Θ , these functions are not uniquely defined by the above equations. The current algebra generated by the momentum and angular momentum densities is now given by:

$$\begin{aligned} \{\tilde{\pi}[\gamma], \tilde{\pi}[\gamma']\} & \approx \int d\sigma \partial_\sigma \gamma'^i (L_{ij}^{(1)}F_{(\gamma)}^j - 2L_{ijk}^{(2)}E_{(\gamma)}^{jk}) , \\ \{\tilde{\mathcal{J}}[\xi], \tilde{\pi}[\gamma]\} & \approx -2\tilde{\pi}[\xi\gamma] + \int d\sigma \partial_\sigma \gamma^i (L_{ij}^{(1)}H_{(\xi)}^j - 2L_{ijk}^{(2)}G_{(\xi)}^{jk}) , \\ \{\tilde{\mathcal{J}}[\xi], \tilde{\mathcal{J}}[\xi']\} & \approx -4\tilde{\mathcal{J}}[\xi\xi'] - 2 \int d\sigma \partial_\sigma \xi'^{ij} \left((x_i L_{kj}^{(1)} - L_{kij}^{(2)})H_{(\xi)}^k + (2x_i L_{j\ell}^{(1)} - \frac{1}{2}L_{ijk\ell}^{(3)})G_{(\xi)}^{k\ell} \right) . \end{aligned} \quad (82)$$

The $\tilde{\pi} - \tilde{\pi}$ and $\tilde{\mathcal{J}} - \tilde{\mathcal{J}}$ Poisson brackets can be shown to satisfy the antisymmetry property using (81).

Once again we get a nontrivial extension Poincaré loop group algebra. Now requiring that all the anomalous terms vanish yields a total of $\frac{1}{8}d(d-1)(d+1)(d+2)$ conditions on the $\frac{1}{8}d(d-1)(d+1)(d+2)$ functions $L_{ij}^{(1)}$, $L_{ijk}^{(2)}$, $L_{ijk\ell}^{(3)}$, and then on the $\frac{1}{8}d(d-1)(d+1)(d+2)$ constants $K_{ij}^{(1)}$, $K_{ijk}^{(2)}$, $K_{ijk\ell}^{(3)}$. As in the previous case, the system is actually underdetermined because the functions $E_{(\gamma)}^{jk}$, $F_{(\gamma)}^k$, $H_{(\xi)}^k$ and $G_{(\xi)}^{jk}$ are not uniquely defined by (81). Again, we have not yet found solutions for the constants K .

5 Generalizations to Membranes

Here we show how to generalize the action for spinning strings to higher dimensional spinning objects, like membranes. The system again contains interesting special cases such as the spinless membrane, where the action is equivalent to the world volume, as well as the case of pure spin, where a spin current is present but the momentum current vanishes. We do not analyze the Hamiltonian dynamics for such systems here.

If, like the case with spinning particles and spinning strings, we let the action depend only on the components V and W of Maurer-Cartan form $g^{-1}dg$, it will automatically be invariant under (left) Poincaré transformations (1). Geometric actions for p -dimensional objects are obtained by constructing p -forms from V and W defined in eq. (5). There are now $p+1$ possible terms:

$$S = S_K(\Lambda, x) = S_1 + S_2 + \dots + S_{p+1}, \quad (83)$$

where

$$S_1 = \int K_{i_1 i_2 \dots i_p}^{(1)} V^{i_1} \wedge V^{i_2} \wedge \dots \wedge V^{i_p}, \quad (84)$$

$$S_2 = \int K_{i_1 j_1 i_2 i_3 \dots i_p}^{(2)} W^{i_1 j_1} \wedge V^{i_2} \wedge \dots \wedge V^{i_p}, \quad (85)$$

.....

$$S_{p+1} = \int K_{i_1 j_1 i_2 j_2 \dots i_p j_p}^{(p+1)} W^{i_1 j_1} \wedge W^{i_2 j_2} \wedge \dots \wedge W^{i_p j_p}, \quad (86)$$

where $K = (K_{i_1 i_2 \dots i_p}^{(1)}, K_{i_1 j_1 i_2 i_3 \dots i_p}^{(2)}, \dots, K_{i_1 j_1 i_2 j_2 \dots i_p j_p}^{(p+1)})$ denotes a set of constants.

Like with the case of strings, the action S_1 by itself describes a spinless object, and was discussed in [4]. This result is easily checked from the corresponding equations of motion.

On the other hand, all the remaining terms give rise to a nonvanishing spin current. The action S_{p+1} by itself has only the spin current contributing to the total angular momentum, as its associated momentum current vanishes.

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